Grey Sets – A Bridge Between Fuzzy Sets and Rough Sets

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Outline for the Presentation

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The necessity for uncertainty models

We are in the Era of Big Data, and we have the chance to either take the Big Data opportunity or suffer Big Data chaos.

- Increased Volume does not mean reduced uncertainty;
- Variety does introduce inconsistence and uncertainty;
- Velocity of Big Data actually requires local small data analysis;
- Veracity requires uncertainty control;

All these show that uncertainty models are essential for the success of Big Data technology, and small data analysis is a necessary part even for Big Data analysis.

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Sources of Uncertainty

- **Objective**
  Unpredictable process;
  Missing information;
- **Subjective**
  Artificial classification;
  Language description;

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Facets of uncertainty

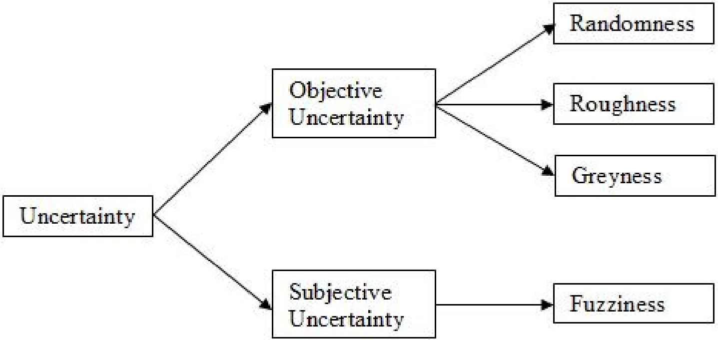

Figure 1: Facets of uncertainty

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Uncertainty models and Computational Intelligence

- **Models for subjective uncertainty**
  Fuzziness – fuzzy sets
- **Models for objective uncertainty**
  Randomness – probability theory
  Roughness – rough sets
  Greyness – grey systems
- **Extended models**
  Interval-valued fuzzy sets
  intuitionistic fuzzy sets
  Type-2 fuzzy sets
  Fuzzy rough sets

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Crisp sets

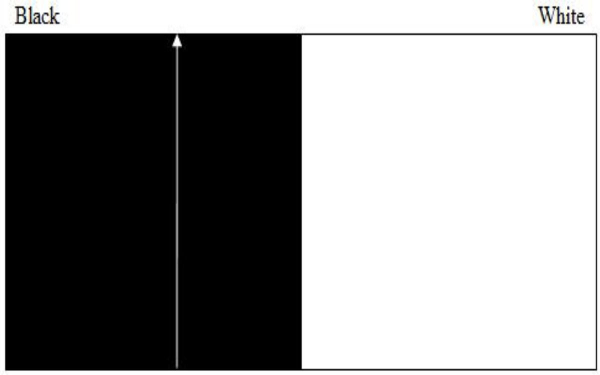

Figure 2: Crisp sets

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Fuzzy sets

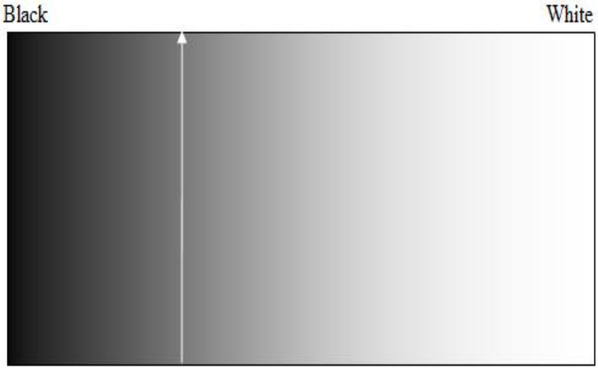

Figure 3: Fuzzy sets

Fuzzy sets

Let $\mathbb{U}$ denote a universe of discourse. Then a fuzzy set A in $\mathbb{U}$ is defined as a set of ordered pairs

$$A = \{\langle x, \mu_A(x)\rangle \mid x \in \mathbb{U}\}$$

where $\mu_A : \mathbb{U} \longrightarrow [0, 1]$ is the membership function of A and $\mu_A(x)$ is the grade of belongingness of x in A.

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Interval-valued fuzzy sets

Let $D[0,1]$ be the set of all closed subintervals of the interval [0,1]. $\mathbb{U}$ is the universe of discourse, x is an element and $x \in X$. An interval-valued fuzzy set in $\mathbb{U}$ is given by set A

$$A = \{\langle x, M_A(x)\rangle \mid x \in \mathbb{U}\}$$

with $M_A : \mathbb{U} \to D[0,1]$

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Intuitionistic fuzzy set

An intuitionistic fuzzy set A in $\mathbb{U}$ is given by

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle \mid x \in \mathbb{U}\}$$

where

$$\mu_A : \mathbb{U} \to [0,1]\ ,\ \ \nu_A : \mathbb{U} \to [0,1]$$

and

$$0 \leq \mu_A(x) + \nu_A(x) \leq 1 \quad \forall x \in \mathbb{U}.$$

For each x, the numbers $\mu_A(x)$ and $\nu_A(x)$ are the degree of membership and degree of non-membership of x in A respectively.

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Type-2 fuzzy sets

A type-2 fuzzy set A, is characterized by a type-2 membership function $\mu_A(x,u)$, where $x \in \mathbb{U}$ and $u \in J_x \subseteq [0,1]$, J_x is a finite discretised set which limits the values of $u \subseteq [0,1]$, i.e.,

$$A = \{\langle (x,u), \mu_A(x,u)\rangle \mid \forall x \in \mathbb{U}, \forall u \in J_x \subseteq [0,1]\}$$

in which $\mu_A : \mathbb{U} \times J_x \longrightarrow [0,1]$. A can also be expressed as

$$A = \int_{x \in \mathbb{U}} \int_{u \in J_x} \mu_A(x,u)/(x,u) \quad J_x \subseteq [0,1]$$

where $\int \int$ denotes union over all admissible x and u. For discrete universes of discourse $\int$ is replaced by $\sum$.

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Approximation

$\Lambda = (\mathbb{U}, A)$ is a given information system, $X \subseteq \mathbb{U}$ is a set. For a given set $B \subseteq A$, the set X is approximated with two sets $B_*(X)$ and $B^*(X)$

$$B_*(X) = \bigcup_{x \in \mathbb{U}} \{B(x) : B(x) \subseteq X\}$$

$$B^*(X) = \bigcup_{x \in \mathbb{U}} \{B(x) : B(x) \cap X \neq \emptyset\}$$

here, $B(x)$ refers to an equivalence class containing x. $B_*(X)$ and $B^*(X)$ are called the B-lower and B-upper approximations of X, respectively.

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Rough sets

Let the pair $apr = (\mathbb{U}, B)$ be an approximation space on $\mathbb{U}$ and $\mathbb{U}/B$ denotes the set of all equivalence classes of B. The family of all definable sets in approximation space apr is denoted by $Def(apr)$. Given two subsets $\underline{A}, \overline{A} \in Def(apr)$ with $\underline{A} \subseteq \overline{A}$, the pair $(\underline{A}, \overline{A})$ is called a rough set.

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Spatial representation of rough sets

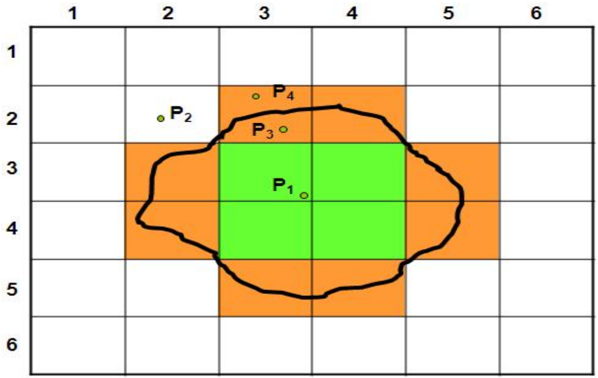

Figure 4: Spatial representation of rough sets

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Fuzzy sets and rough sets

- A fuzzy set represents a set with its membership values, but a rough set approximates a set through partitions.
- Fuzzy sets focus on the vagueness of the boundary between a set and its complement, but rough sets highlight the incomplete information in a given information system (table).
- Therefore, they provide models for two different facets of uncertainties.

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Fuzzy rough sets

Fuzzy rough sets are defined by membership function on the universe of objects U by

$$\mu_{S^*(X)}(x) = \sup_{w \in X} \mu_S(x, w)$$

$$\mu_{S_*(X)}(x) = \inf_{w \notin X} (1 - \mu_S(x, w))$$

where S is a fuzzy similarity relation and $x \in U$.

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Grey systems

Figure 5: Grey systems

Grey numbers

- A grey number is a number with clear upper and lower boundaries but which has an unknown position within the boundaries
- It is expressed as

$$a^{\pm} \in [a^{-}, a^{+}] = \{t \in a^{\pm} | a^{-} < t < a^{+}\}$$

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Discrete grey numbers can not be described using interval

- **A grey number g can only select one value from 12, 15 and 18**
  $g = 12$ or $g = 15$ or $g = 18$
  There are no other options
  The probability for each number to be the white number is $\frac{1}{3}$
- **If we write it as $g = [12, 18]$**
  g could be anything between 12 and 18
  12 is possible
  12.0099 is possible
  ...
- **What would be their probability ?**

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Limitation of interval operations

- Each number inside $a^{\pm}$'s boundary $[a^-, a^+]$ would be an eligible candidate for the underlying white number
- the probability for a number $a^- \leq b_i \leq a^+$ to be the underlying white number is: $P(b_i) = \frac{1}{\infty} \to 0$
- If we know the underlying white number can only be one of limited candidates $\{b_1, b_2, \ldots, b_n\}$ $(n < \infty)$, the real probability should be: $P(b_i) = \frac{1}{n}$
- The operations for discrete grey numbers are totally different from those for continuous grey numbers, and then can not be derived directly from interval calculus

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Definition

Let $v \in \Re$ be a real number and $v^{\pm}$ be a union set of closed or open intervals

$$v^{\pm} \in \bigcup_{i=1}^{n} [a_i^-, a_i^+] \tag{1}$$

Here $i = 1, 2, \ldots, n$, n is an integer and $0 < n < \infty$, $a_i^-, a_i^+ \in \Re$ and $a_{i-1}^+ \leq a_i^- \leq a_i^+ \leq a_{i+1}^-$. For any interval $[a_i^-, a_i^+] \in v^{\pm}$, p_i is the probability for $v \in [a_i^-, a_i^+]$. If the following conditions hold for

- $p_i > 0$
- $\sum_{i=1}^{n} p_i = 1$

then we call v a grey number represented by $v^{\pm}$. $v^- = \inf_{a_i^- \in v^{\pm}} a_i^-$ and $v^+ = \sup_{a_i^+ \in v^{\pm}} a_i^+$ are called as the lower and upper limits of $v^{\pm}$.

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Remarks

- It is clear that a grey number $v^{\pm}$ is different from the set $v^{\pm}$
- The grey number $v^{\pm}$ represents only one number which is not clearly identified among the elements in set $v^{\pm}$
- For any grey number $v^{\pm} \in \Re$, if $|v^{+} - v^{-}| = 0$, then this $v^{\pm}$ is called a white number
- For any grey number $v^{\pm} \in \Re$, if $|v^{+} - v^{-}| = \infty$, then this $v^{\pm}$ is called a black number

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Theorem

$v^{\pm}$ is a grey number. The following properties hold for $v^{\pm}$:

- $v^{\pm}$ is a continuous grey number $v^{\pm} \in [a_1^-, a_n^+]$ iff $a_i^- = a_{i-1}^+$ $(\forall i > 1)$ or $n = 1$
- $v^{\pm}$ is a discrete grey number $v^{\pm} \in \{a_1, a_2, \ldots, a_n\}$ iff $a_i = a_i^- = a_i^+$
- $v^{\pm}$ is a mixed grey number iff only part of its intervals shrink to crisp numbers and others keep as intervals.

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Degree of greyness of a grey number

For any grey number $v^{\pm}$, $v^{-}, v^{+} \in \Re$ are its lower and upper limits. The degree of greyness of $v^{\pm}$ is a function $g^{\circ} = f(v^{-}, v^{+})$ satisfying the following conditions

- $g^{\circ} \geq 0$
- $g^{\circ} = 0$ iff $v^{-} = v^{+}$

There are many functions satisfying these conditions, and hence there may be many different valid functions for degree of greyness of grey numbers. If we consider grey numbers in the positive $\Re^{+} \cup \{0\}$, we can define it as

$$g^{\circ} = \frac{v^{+} - v^{-}}{v^{+} + v^{-}}$$

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Remarks

- The degree of greyness of a grey number depends only on the two limits of a grey number and has nothing to do with the cardinality of its candidate set
- The degree of greyness is a parameter for the grey number rather than any candidate in its candidate set

For example, both ${v_1}^{\pm} \in [40, 60]$ and ${v_2}^{\pm} \in \{40, 60\}$ have the same degree of greyness

$$ {g_1}^{\circ} = {g_2}^{\circ} = \frac{60 - 40}{60 + 40} = 0.2 $$

but their cardinalities are completely different:

$$ Card({g_1}^{\pm}) = \infty, \quad Card({g_2}^{\pm}) = 2 $$

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Arithmetic operations of extended grey numbers

$$a^{\pm} + b^{\pm} \in \bigcup_{i=1}^{m} \bigcup_{j=1}^{n} [a_i^- + b_j^-, a_i^+ + b_j^+] \tag{2}$$

$$a^{\pm} - b^{\pm} \in \bigcup_{i=1}^{m} \bigcup_{j=1}^{n} [a_i^- - b_j^+, a_i^+ - b_j^-] \tag{3}$$

$$a^{\pm} \times b^{\pm} \in \begin{array}{l} \bigcup_{i=1}^{m} \bigcup_{j=1}^{n} [\min\{a_i^- b_j^-, a_i^+ b_j^+, a_i^- b_j^+, a_i^+ b_j^-\}, \\ \max\{a_i^- b_j^-, a_i^+ b_j^+, a_i^- b_j^+, a_i^+ b_j^-\}] \end{array} \tag{4}$$

$$\frac{a^{\pm}}{b^{\pm}} \in \begin{array}{l} \bigcup_{i=1}^{m} \bigcup_{j=1}^{n} [\min\{\frac{a_i^-}{b_j^-}, \frac{a_i^+}{b_j^+}, \frac{a_i^-}{b_j^+}, \frac{a_i^+}{b_j^-}\}, \\ \max\{\frac{a_i^-}{b_j^-}, \frac{a_i^+}{b_j^+}, \frac{a_i^-}{b_j^+}, \frac{a_i^+}{b_j^-}\}] \end{array} \tag{5}$$

$$b^{\pm^{-1}} \in \bigcup_{j=1}^{n} \left[\frac{1}{b_j^+}, \frac{1}{b_j^-}\right] \tag{6}$$

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Combined uncertainty

- In real world, the objective uncertainty and subjective uncertainty may exist in the same time.
- For example, we may not know the exact membership value of a fuzzy set, but we know its value coming from a set of values.
- There have been many extensions to fuzzy sets, such as interval-valued fuzzy sets, intuitionistic fuzzy sets, type-2 fuzzy sets, R-fuzzy sets and etc.
- Obviously, a grey number is an ideal model when the exact value of a membership is not completely known.

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White sets and black set

Let U be the finite universe of discourse, $x \in U$ and $A \subseteq U$.

- If the characteristic function value of each x with respect to A can be expressed with a single white number $v \in [0, 1]$

$$\chi_A : U \to [0, 1]$$

  then A is a white set
- If the characteristic function value of each x with respect to A can only be expressed with a grey number $b^{\pm}$ with $b^{-} = 0$ and $b^{+} = 1$, then A is a black set.

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Grey sets

- For a set $A \subseteq U$, if the characteristic function value of x with respect to A can be expressed with a grey number $v^{\pm}$

$$\chi_G : U \to D[0,1]^{\pm}$$

  then A is a grey set.
- Expression
  Similar to the expression of a fuzzy set, a grey set A is represented with its relevant elements and their associated grey number for characteristic function:

$$A = {v_1}^{\pm}/x_1 + {v_2}^{\pm}/x_2 + \ldots + {v_n}^{\pm}/x_n$$

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Remarks

- It should be noted that the grey numbers representing the characteristic function values are grey numbers rather than intervals
- A related extension of fuzzy sets is interval-valued fuzzy sets. The membership value of an element belonging to an interval-valued fuzzy set is defined as an interval.
- Some recent research publications called grey sets as grey fuzzy sets where the membership values are intervals. Obviously, such a definition makes grey sets no difference from interval-valued fuzzy sets.
- From our analysis of grey numbers, it is clear that they are differrent.

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Example table

Name	Gender	Working Attitude	Exam Result
Mike	Male	Good	Good
Jane	Female	Neutral	Good
Claire	Female	Neutral	Neutral
David	Male	Neutral	Poor
Lisa	Female	Poor	Poor

Table 1: Information for 5 people

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Example result

The exam result shows some kind of relationships with Working Attitude. A characteristic function is established according to the relationship between Working Attitude and Exam Result:

$$f_A{}^c(a_i) = \begin{cases} 1 & \text{if } a_i\text{'s Working Attitude= good;} \\ \{0, 0.5, 1\} & \text{if } a_i\text{'s Working Attitude = unknown;} \\ 0 & \text{if } a_i\text{'s Working Attitude = poor.} \end{cases}$$

Under this characteristic function,

$$\begin{aligned} A &= \frac{[1,1]}{Mike} + \frac{\{0,0.5,1\}}{Jane} + \frac{\{0,0.5,1\}}{Claire} + \frac{\{0,0.5,1\}}{David} + \frac{[0,0]}{Lisa} \\ &= \frac{1}{Mike} + \frac{\{0,0.5,1\}}{Jane} + \frac{\{0,0.5,1\}}{Claire} + \frac{\{0,0.5,1\}}{David} + \frac{0}{Lisa} \end{aligned}$$

Obviously, A is a grey set rather than an interval-valued fuzzy set.

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Grey elements

U is the universe of discourse, A is a grey set and $A \subseteq U$. x is an element relevant to A and $x \in U$. $v^{\pm}$ is the value for characteristic function of x with respect to A.

- **White elements** If $v^{-} = v^{+}$, then x is called a white element
- **Black elements** If $v^{-} = 0$ and $v^{+} = 1$, then x is called a black element
- **Grey elements** If $v^{-} \neq v^{+}$, then x is called a grey element

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Degree of greyness for grey elements

The degree of greyness $g^{\circ}{}_{A}(x)$ of element x for set $A \subseteq U$ is expressed as

$$g^{\circ}{}_{A}(x) = \frac{v^{+} - v^{-}}{v^{max} - v^{min}}$$

Here,v^{max} and v^{min} are the maximum value and minimum value of characteristic function.
According to our definition for grey sets, we have

$$g^{\circ}{}_{A}(x) = v^{+} - v^{-}$$

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Degree of greyness for grey sets

U is the finite universe of discourse, A is a grey set and $A \subseteq U$. x_i is a an element relevant to A and $x_i \in U$. $i = 1, 2, 3, \ldots, n$ and n is the cardinality of A. The degree of greyness of set A is defined as

$$g^{\circ}{}_{A} = \frac{\sum g^{\circ}{}_{A}(x_i)}{n}$$

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Example results for grey sets

Name	Exam Result		Working Attitude		Gender	
	Element	Set	Element	Set	Element	Set
Mike	0		0		1	
Jane	0		1		1	
Claire	0	0	1	0.6	1	1
David	0		1		1	
Lisa	0		0		1	

Table 2: Example for degree of greyness

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Relationship with crisp sets and fuzzy sets

U is the finite universe of discourse, A is a grey set and $A \subseteq U$. x is an element and $x \in U$. $v^{\pm}{}_x$ is a value for the characteristic function with respect to x. $g^{\circ}{}_G(x)$ is the degree of greyness of x, and $g^{\circ}{}_G$ is the degree of greyness for A. The following properties hold for x and A:

- A is a white set iff $g^{\circ}{}_G = 0$
- A is a black set iff $g^{\circ}{}_G = 1$
- A is a crisp set iff $g^{\circ}{}_G = 0$ and $v^{\pm}{}_x \in \{0, 1\}$ for any $x \in U$
- A is a type-1 fuzzy set iff $g^{\circ}{}_G = 0$ and $v^{\pm}{}_x \in [0, 1]$ for any $x \in U$
- A is an interval-valued fuzzy set iff $v^{\pm}{}_x$ is a continuous grey number for any $x \in U$

This theorem shows that grey sets extend crisp sets, fuzzy sets and interval-valued fuzzy sets

Relationship with rough sets

- Both grey sets and rough sets are based on incomplete information.
- When enough information is available, then a grey set is turned into a white set, and a rough set is turned into a crisp set.
- The difference between the resulted white set and crisp set is that the white set might be a fuzzy set.
- If we remove fuzziness from a grey set, it is actually equivalent to a rough set.

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Relationship with rough sets

We have the following theorem:

A is a rough set iff ${g^\circ}_A > 0$ and ${v^\pm}_x \subseteq \{0, 1\}$ holds for any $x \in U$. Here, U is the finite universe of discourse, A is a grey set and $A \subseteq U$. x is an element and $x \in U$. ${v^\pm}_x$ is a value for the characteristic function with respect to x. ${g^\circ}_A$ is the degree of greyness for A.

This theorem proves that grey sets include rough sets as a special case.

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A bridge between fuzzy sets and rough sets

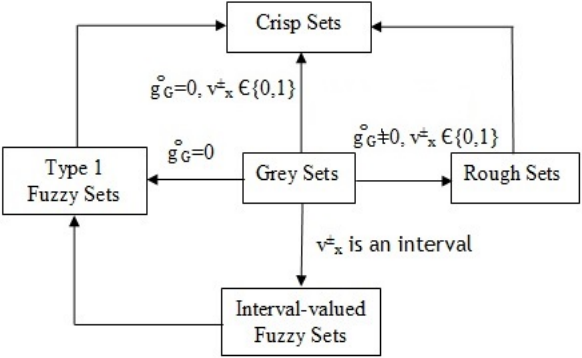

Figure 6: The relationship between different sets

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Conclusions

- Fuzziness and roughness represent different aspects of uncertainties, and provide complementary functions in uncertainty modelling.
- Greyness overlaps with both fuzziness and roughness, and provide an alternative route to unify fuzzy sets and rough sets.
- We propose grey sets unifying fuzzy sets and rough sets in a simple model. Our results show that a grey model can integrate both fuzziness and roughness in one model.

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Thanks for your attention !

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