Grey-Based Preference in a Graph Model for Conflict Resolution With Multiple Decision Makers

Hanbin Kuang, Member, IEEE, M. Abul Bashar, Keith W. Hipel, Fellow, IEEE, and D. Marc Kilgour

Abstract—To capture uncertainty in preferences, definitions based on grey numbers are incorporated into the graph model for conflict resolution (GMCR), a realistic and flexible methodology to model and analyze strategic conflicts. A general grey number is a real number that may be a member of a discrete set of real numbers, or may fall within one or several intervals. It can represent uncertain preference of decision makers in a meaningful way. Here, a grey-based preference structure is developed and integrated with GMCR. Utilizing a number of grey-based ideas, solution concepts describing human behavior under conflict in the face of uncertain preference are defined for a conflict model. This grey-based GMCR is then applied to a generic sustainable development conflict with uncertain preferences in order to demonstrate how it can be conveniently utilized in practice.

Index Terms—Decision analysis, general grey numbers, graph model for conflict resolution (GMCR), sustainable development, uncertain preference.

I. INTRODUCTION

DECISION analysis techniques and methodologies have been designed for tackling highly structured problems using quantitative mathematical techniques. However, in some situations, the solution objective for a decision problem may not be optimal within a well-defined structure, but satisfaction of a decision maker (DM) under complex circumstances, in which qualitative criteria and interactions with other DMs must be considered [1]. At the strategic level, decision problems are frequently complicated and not well-defined.

When two or more DMs are involved in a decision situation, a conflict may arise as the DMs interact with others to further their own interests, which may be different [2], [3]. Each DM may have his or her own criteria to determine preferences among the possible scenarios. Hence, each DM may have a separate multiple criteria decision problem to rank scenarios. Strategic conflicts are interactive decision problems, in which each DM controls one or more options and attempts to achieve the most preferable scenario for himself or herself. Note that if one DM exercises an option, it may benefit or harm the other DMs. Therefore, cooperative or compromise solutions may be available [4]. In practice, these phenomena arise frequently in many circumstances, such as military strategy, business negotiation, and environmental decision making [5]–[8].

The graph model for conflict resolution (GMCR) has proven to be a simple and flexible methodology to model and analyze strategic conflicts. Much valuable research has been conducted regarding different aspects of this methodology with respect to both theory and practice. Fang et al. [9] focused on solution concepts and their interrelationship, while Hipel et al. [10] explained the roles of GMCR and other operational research tools to solve problems within a systems engineering context. Kilgour and Hipel [4] reviewed various initiatives within the GMCR framework and suggested guidelines for future development. To implement the graph model methodology, a user-friendly decision support software, GMCR II, has been developed [11]. It can reliably model and analyze multiple participant-multiple objective conflicts, small or large [12].

In classic GMCR, it is assumed that each DM can rank all the scenarios in a model, and stability analysis is carried out based on these preference rankings. However, in some situations, limitations of human cognition, incomplete information, interplay of DMs, and the complexity, may make a DM unclear or uncertain about his preference over possible scenarios [13]. Several studies have been conducted to address group decision making with uncertain preference information. Han et al. [14] concentrated on modeling grey conflicts based on grey input data. Ben-Haim and Hipel [15] used the information-gap model to estimate the strategic impacts of uncertain preference for DMs. Li et al. [16] put forward an uncertain preference relation—“preferred to,” “indifferent to,” and “unknown”—in modeling preference uncertainty in the graph model. Bashar et al. [17] and Hipel et al. [18] developed a methodology to model and analyze a conflict with fuzzy preference. Bristow et al. [19] proposed a systems methodology to simulate multiple participants’ responses in a dynamic environment.

A new preference structure based on general grey numbers is introduced in this paper. A generalized grey number with values ranging from 0 to 1, may represent a preference degree,
interval of preference degrees, or combinations thereof, and can be used to capture preference uncertainty between two states. This approach allows DMs to describe preferences with generalized grey numbers, in addition to strict preference and indifference, thereby implementing a new form of possibilistic reasoning.

The research contained in this paper constitutes a significant expansion of earlier work presented in a conference paper by Kuang et al. [20]. In [20], grey numbers are incorporated into GMCR, and a case study is provided to illustrate the feasibility of using grey numbers to represent the uncertain preferences of DMs. However, in this paper, the formal mathematical definitions for grey preferences and grey stability definitions in GMCR are not provided. This research, using grey numbers to express uncertain preferences of DMs, formally puts forward grey preferences, and for the first time defines grey-based stability concepts within the GMCR structure, thereby extending the graph model methodology. Corresponding grey-based equilibria can then be identified, indicating more realistic resolutions for conflicts in the face of uncertainty.

The main contributions of this approach within the framework of conflict analysis are shown in Fig. 1, which is modified from Fang et al. [9] and Fraser and Hipel [21]. Modeling a real-world conflict consists of identifying the DMs, their options, and their relative preferences over states. This paper extends the classical GMCR by mathematically defining grey preferences, grey stabilities, and grey equilibria. Moreover, interpretation and sensitivity analyses are carried out using a sustainable development conflict. The results can be convincingly used to model the conflict.

![Graph Model Methodology](image)

**Fig. 1.** Main contributions within the framework of conflict analysis.

II. BASIC METHODOLOGIES

In the following subsections, GMCR with two DMs and GMCR with multiple DMs are defined along with solution concepts for carrying out a stability analysis. Next, ideas from grey system theory are defined before introducing grey preferences and grey stability.

A. GMCR With Two DMs

The main purpose of the graph model methodology is to describe the key characteristics of a conflict based on the possible interactions of DMs as dictated by their strategies and preferences, and then generate the possible compromise resolutions, or equilibria, through extensive analyzes [12]. The graph model methodology represents possible scenarios (or states) of a conflict as vertices of a graph, and the transitions controlled by each DM as the arcs of the graph, labeled by the DM controlling a given move in one step. Note that, in a graph model, the movements of DMs can be reversible or irreversible, and no loops are contained in any DM’s graph. Within the graph model paradigm, a conflict model requires four main components: 1) a set of DMs; 2) a set of feasible states; 3) possible movements between states in one step controlled by each DM; and 4) each DM’s relative preference over the feasible states [9]. To formally model a conflict, the four fundamental components of GMCR are mathematically expressed as follows [9], [12].

1) \( N = \{1, 2, \ldots, n\}, n \geq 2 \), represents the set of DMs. For the case of two DMs, \( N = \{1, 2\} \).
2) \( S = \{s_1, s_2, \ldots, s_m\}, m > 1 \), denotes the set of feasible states. The particular state where the conflict begins is designated as the status quo state.
3) For \( k \in N, G_k = (S, A_k) \) denotes DM \( k \)’s directed graph. Here, \( A_k \subseteq S \times S \) represents the set of arcs controlled by DM \( k \). For example, for \( s_i, s_j \in S \) and \( s_i \neq s_j \), \( (s_i, s_j) \in A_k \) if and only if DM \( k \) can unilaterally move the conflict from state \( s_i \) to state \( s_j \) in one step. In this case, state \( s_j \) is reachable from state \( s_i \) for DM \( k \). Then, the collection of the directed graphs of all the DMs, \( G = \{(S, A_k), k \in N\} \), can be convincingly used to model the conflict.
4) For \( k \in N \), a binary relation \( \succ_k \) on \( S \) expresses DM \( k \)’s preference of one state over another. Specifically, for \( s_i, s_j \in S \), \( s_i \succ_k s_j \) indicates that \( s_i \) is strictly preferred to \( s_j \) by DM \( k \), while \( s_i \sim_k s_j \) means that \( s_i \) is equally preferred to \( s_j \).

B. STABILITY DEFINITIONS FOR CONFLICT WITH TWO DMs

In general, if a DM has no incentive to move from the present state, this state is stable for the DM. If a state is stable for all DMs, it constitutes an equilibrium of the model. The main stability definitions included in the graph model are Nash stability (R) [22], general metarationality (GMR) [23], symmetric metarationality (SMR) [23], and sequential stability (SEQ) [21]. These stability definitions incorporate moves and countermoves as DMs attempt to do as well as possible, thereby representing common patterns of human behavior under conflict.

To formally define the previously mentioned four major stability concepts within the graph model framework, one needs to identify states that are unilaterally reachable by a DM.
Accordingly, the definitions of reachable list and the unilateral improvement list are given below [4].

**Definition 1:** Let \( k \in N \) and \( s \in S \). The reachable list from state \( s \) for DM \( k \) is

\[
R_k(s) = \{ s_i \in S : (s, s_i) \in A_k \}.
\]  

(1)

The reachable list from a given state for a DM represents a collection of all possible states to which the DM can move in one step.

**Definition 2:** Let \( k \in N \) and \( s \in S \). The unilateral improvement list from state \( s \) for DM \( k \) is

\[
R_k^+(s) = \{ s_i \in R_k(s) : s_i >_k s \}.
\]  

(2)

The unilateral improvement list from a given state for a DM is the collection of all preferred states, compared with the given state, to which the DM can unilaterally move.

In GMCR, if the focal DM has no incentive to move from an initial state, this state is stable for him. Stability definitions (or solution concepts) were introduced to identify such states [4]. Let \( N = \{ p, q \} \) represent the set of DMs in a conflict having two DMs, given by \( p \) and \( q \). Then, a brief summary of the four stability definitions mentioned above in a two-DM conflict within the framework of graph model is presented as follows [9], [11], [12].

**Definition 3:** A state \( s \in S \) is Nash stable (or rational) for DM \( p \), denoted by \( s \in S_p^N \), if and only if \( R_p^+(s) = \emptyset \).

From a Nash stable state, the focal DM has no unilateral improvement to which to move. Thus, from a given state, if there is a unilateral improvement, the DM will move to it. For this stability type, the focal DM does not take into account possible responses by his opponent. However, a DM may consider possible countermoves if he moves to an advantageous state. The next three definitions characterize rules to identify stable states for a DM with foresight.

**Definition 4:** A state \( s \in S \) is GMR for DM \( p \), denoted by \( s \in S_p^G \), if and only if for every \( s_1 \in R_p^+(s) \) there exists at least one \( s_2 \in R_p(s_1) \) such that \( s_2 >_p s \).

In GMR stability, it is assumed that any unilateral improvement from the initial state for the focal DM can be sanctioned by a subsequent unilateral movement by the other DM. In this situation, the initial state is GMR stable.

**Definition 5:** A state \( s \in S \) is SMR stable for DM \( p \), denoted by \( s \in S_p^S \), if and only if for every \( s_1 \in R_p^+(s) \) there exists at least one \( s_2 \in R_p(s_1) \) such that \( s_2 >_p s \). The case that \( s_2 <_p s \) for all \( s_3 \in R_p(s_2) \).

In SMR stability, any unilateral improvement for the focal DM from the initial state can be sanctioned by a subsequent unilateral move of the other DM, and the focal DM cannot escape the sanction through another countermove. In this case, the initial state is SMR.

**Definition 6:** A state \( s \in S \) is sequentially stable for DM \( p \), denoted by \( s \in S_p^S \), if and only if for every \( s_1 \in R_p^+(s) \) there exists at least one \( s_2 \in R_p^+(s_1) \) such that \( s_2 <_p s \).

The definition of SEQ is the same as GMR stability except that while considering the sanction of the focal DM’s unilateral improvement imposed by the opponent, the focal DM only takes into account the opponent’s unilateral improvements rather than unilateral movements.

**C. GMCR With Multiple DMs**

For a conflict involving more than two DMs, it is necessary to define joint moves. The reachable list for multiple DMs from a given state represents a collection of all possible states to which some or all of the DMs can move via a legal sequence of moves, in which the same DM may move more than once, but not twice consecutively.

**Definition 7:** Let \( s \in S, H \subseteq N \), and \( |H| \geq 2 \). Let \( \Omega_H(s, s) \) denote the set of all last DMs in legal sequences of unilateral moves from \( s \) to \( s_j \). The reachable list \( R_H(s) \) for \( H \) from state \( s \) is defined inductively as follows.

1) If \( k \in H \) and \( s_1 \in R_k(s) \), then \( s_1 \in R_H(s) \) and \( k \in \Omega_H(s, s_1) \).
2) If \( s_1 \in R_H(s), k \in H, s_2 \in R_k(s_1) \), and \( \Omega_H(s, s_1) \neq \{ k \} \), then \( s_2 \in R_H(s) \) and \( k \in \Omega_H(s, s_2) \).

Note that the definition stops only when no new state can be added to \( R_H(s) \). The unilateral movement list is the collection of all unilateral movements for any nonempty subset of the DMs from the given state.

**Definition 8:** Let \( s \in S, H \subseteq N, \) and \( H \geq 2 \). Let \( \Omega_H^+(s, s) \) denote the set of all last DMs in legal sequences allowable for implementing a unilateral improvement from \( s \) to \( s_j \). Then, the unilateral improvement list \( R_H^+(s) \) for state \( s \) for \( H \) is defined inductively as follows.

1) If \( k \in H \) and \( s_1 \in R_H^+(s) \), then \( s_1 \in R_H^+(s) \) and \( k \in \Omega_H^+(s, s_1) \).
2) If \( s_1 \in R_H^+(s), k \in H, s_2 \in R_H^+(s_1) \), and \( \Omega_H^+(s, s_1) \neq \{ k \} \), then \( s_2 \in R_H^+(s) \) and \( k \in \Omega_H^+(s, s_2) \).

Keep in mind that the definition stops only when there is no more new state. A joint unilateral improvement from a given state by a subset of at least two DMs is a state that is in the reachable list for these DMs from the initial state and worthwhile for all of them to move to. Specifically, if a group of DMs, \( H \), moves the conflict from state \( s_1 \) to \( s_2 \) via a legal sequence of moves and each movement is a unilateral improvement for one DM, then \( s_2 \) is a unilateral improvement for \( H \), as are the intermediate states. The unilateral improvement list is the collection of all unilateral improvements from the given state for any nonempty subset of the DMs.

**D. Stability Definitions for Conflict With n-DMs (n ≥ 2) DMs**

In a conflict with more than two DMs, the opponent of a focal DM is a group (or coalition) of DMs. Taking into account possible subsequent movements of the group of other DMs, if the focal DM has no initiative to move from an initial state, this state is stable for him or her. Stability definitions (or solution concepts) are introduced to identify such states [4]. Assume that \( S = \{ s_1, s_2, \ldots, s_m \} \), \( m > 1 \) denotes the set of feasible states, and \( N \) represents the set of DMs. Then, a brief summary of four stability definitions in an \( n \)-DM (\( n > 2 \)) conflict is presented as follows [11], [12].

**Definition 9:** Let \( k \in N \), a state \( s \in S \) is Nash stable (or rational) for DM \( k \), denoted by \( s \in S_p^R \), if and only if \( R_k^+(s) = \emptyset \).

For a Nash stable state, the focal DM has no unilateral improvement from the initial state. The definition is the same as the model with two DMs.
Definition 10: Let \( k \in N \) and \( N - \{k\} \) denote all the other DMs except \( k \). A state \( s \in S \) is GMR for DM \( k \), denoted by \( s \in S^GMR_k \), if and only if for every \( s_1 \in R^k_s \) there exists at least one \( s_2 \in R_{N-\{k\}}^s(s_1) \) such that \( s_2 \succeq s \).

In GMR stability for multiple DMs, any unilateral improvement for the focal DM from the initial state can be sanctioned by subsequent sequential unilateral movements by other DMs.

Definition 11: Let \( k \in N \) and \( N - \{k\} \) denote all the other DMs except \( k \). A state \( s \in S \) is SMR stable for DM \( k \), denoted by \( s \in S^{SMR}_k \), if and only if for every \( s_1 \in R^k_s \) there exists at least one \( s_2 \in R_{N-\{k\}}^s(s_1) \) such that \( s_2 \preceq s \), \( s_3 \preceq s \) for all \( s_3 \in R^k_s(s_2) \).

In SMR stability for multiple DMs, the focal DM’s unilateral improvements are sanctioned by subsequent sequential unilateral moves by other DMs, and the focal DM cannot escape the sanction through another countermove.

Definition 12: Let \( k \in N \), and \( N - \{k\} \) denote all the other DMs except \( k \). A state \( s \in S \) is sequentially stable for DM \( k \), denoted by \( s \in S^{SEQ}_k \), if and only if for every \( s_1 \in R^k_s \) there exists at least one \( s_2 \in R_{N-\{k\}}^s(s_1) \) such that \( s_2 \preceq s \).

In SEQ, the focal DM’s unilateral improvement from the initial state can be sanctioned by subsequent unilateral improvements by other DMs.

The four stability definitions for the \( n \)-DM (\( n > 2 \)) graph model provided above indicate that opponents of the focal DM are multiple DMs. In considering moving to possible unilateral improvements, the focal DM needs to take account of sanctions, which can be imposed by multiple DMs through subsequent sequential unilateral movements or unilateral improvements.

The relationships among the four stability definitions provided above are explained in Table I. These stability concepts are defined based on the characteristics of DMs, and can describe DMs’ different reactions when they are dealing with potential risks. A DM with no foresight will follow R, and he ignores risks and chooses any possible strategy to gain benefit. However, GMR, SMR, and SEQ describe DMs with foresight, and they will take the countermoves of the opponent(s) into account. A DM who follows GMR or SMR expects to be sanctioned by the opponent(s) at any cost. Hence, the DM is conservative and tries to avoid risks. On the contrary, a DM who follows SEQ believes that he will be sanctioned by the opponent(s) without sacrificing his own benefits. He may take some risks [4], [24].

An issue in the application of the graph model is the difficulty in ascertaining DMs’ preferences. Other methodologies that permit uncertain preferences in a graph model have been proposed, such as unknown [16] and fuzzy [17], [18] preferences. In an effort to develop a maximally flexible representation of preference uncertainty, this paper introduces grey preferences and shows how they can be incorporated into the graph model.

E. Grey System Theory

Grey system theory, originally put forward by Deng in 1982 [25], is a methodology that focuses on addressing problems with imperfect numerical information, which may be discrete or continuous [26], [27]. In grey system theory, a system with information that is certain is called a white system; a system with information that is totally unknown is referred to as a black system; a system with partially known and partially unknown information is called a grey system. The theory contains five main parts: 1) grey prediction models; 2) grey relational analysis; 3) grey models for decision making; 4) grey game models; and 5) grey control systems [27].

The methodologies of grey system theory can effectively handle representation and processing of vague or uncertain information, and they can provide insights into operational features of systems and their evolution.

Grey system theory is receiving increasing attention in the field of decision making, and has been successfully applied to many important problems featuring uncertainty [28]. Chan and Tong [29] used grey relational analysis in multiple criteria material selection. Li et al. [30] developed a grey-based approach to deal with supplier selection. Özcan et al. [31] made a comparison among various multiple criteria decision analysis methods and grey relational analysis, and then applied grey relational analysis to a warehouse selection problem. Liou et al. [32] provided another application aimed at improving airline service quality. During the last decade, increasing applicability of grey system theory motivated many researchers to compare it with related techniques and invent new combinations. Zhang et al. [33] investigated grey related analysis using fuzzy interval numbers. Wei [34] extended grey system theory to investigate intuitionistic fuzzy multiple attributes decision problems; and Tseng [35] combined linguistic preferences with grey relational analysis in fuzzy environmental management.

A crucial feature of a method for solving decision problems under uncertainty is how it deals with uncertain information [36]. Grey system theory, in particular, possesses many desirable characteristics. Firstly, it can handle both quantitative and qualitative data. Secondly, grey numbers can represent not only one or more discrete values but also intervals of numbers, depending on the DMs’ opinions. Thirdly, grey system theory uses the concept of degree of greyness to estimate the uncertainty of grey numbers, rather than a typical distribution of their values. However, most researchers and practitioners focus on grey relational analysis and its combination with other techniques, with little attention paid to theoretical foundations and extensions of definitions of grey numbers and associated theorems to make them more suitable to represent uncertain preferences of DMs [37], [38]. In this subsection, the definitions of grey numbers and their calculation rules are presented.

A grey number is the most fundamental concept in grey system theory. In the original definitions, a white number is a real number, \( x \in \mathbb{R} \). A grey number, written \( \pounds x \), means an indeterminate real number that takes its possible values.

<table>
<thead>
<tr>
<th>Solution Concepts</th>
<th>Foresight</th>
<th>Sanctions</th>
<th>Strategic Risk</th>
</tr>
</thead>
<tbody>
<tr>
<td>R</td>
<td>No</td>
<td>Ignore</td>
<td></td>
</tr>
<tr>
<td>GMR</td>
<td>2</td>
<td>Unilateral Moves</td>
<td>Avoid</td>
</tr>
<tr>
<td>SMR</td>
<td>3</td>
<td>Unilateral Moves</td>
<td>Avoid</td>
</tr>
<tr>
<td>SEQ</td>
<td>2</td>
<td>Unilateral Improvements</td>
<td>Take Risks</td>
</tr>
</tbody>
</table>

TABLE I
SOLUTION CONCEPTS IN A GRAPH MODEL
(MODIFIED FROM FANG ET AL. [24])
Fig. 2. Distinguishing grey numbers from probability distributions and fuzzy numbers.

within an interval or a discrete set of numbers. The following definitions from [37], Yang and John generalize definitions proposed earlier in [27].

Definition 13: A general grey number, \( \otimes x \), is a real number that is not known but has a clear lower bound and an upper bound, \( x \) and \( \bar{x} \in \mathbb{R} \), respectively, taking its value from the closed interval, \( [x, \bar{x}] \), denoted as [37]

\[
\otimes x \in \bigcup_{i=1}^{k} [x_i, \bar{x}_i] 
\]

where \( 1 \leq k < \infty \), \( x_j, \bar{x}_i \in \mathbb{R} \), and \( \bar{x}_{i-1} < x_j \leq \bar{x}_i < \bar{x}_{i+1} \), \( x = \min x_j \), and \( \bar{x} = \max \bar{x}_i \).

Equation (3) constitutes a very general definition. A grey number is a real number that has a precise lower and upper bound, but its position between the lower and upper bounds is not known. It is a real number which may be a member of a discrete set of real numbers, may fall within an interval of real numbers, or reside within any combination of intervals and discrete sets. Some illustrations of grey numbers are as follows.

1) If \( x_j = \bar{x}_i = x_i \) for all \( i = 1, 2, \ldots, k \), then \( \otimes x \in \{x_1, x_2, \ldots, x_k\} \) is a grey number (a member of a discrete set).

2) If \( k = 1 \) and \( x \neq \bar{x} \), then \( \otimes x \in [x, \bar{x}] \) is a grey number (a real number residing within an interval).

3) If \( x_j = \bar{x}_i = x \) and \( k = 1 \), then \( \otimes x = x \in \mathbb{R} \) is a grey number.

Example 1: Three grey numbers, \( \otimes x_1 \in \{0.1, 0.2, 0.3\}, \otimes x_2 \in [0.2, 0.4] \), and \( \otimes x_3 \in \{0.1, 0.2, 0.3, 0.5\} \) constitute a discrete set of real numbers, an interval, and a union of intervals, respectively.

Let \( \otimes x_1 \) and \( \otimes x_2 \) be two grey numbers, \( \otimes x_1 \in \bigcup_{i=1}^{m} [x_i, \bar{x}_i], \otimes x_2 \in \bigcup_{j=1}^{n} [x_j, \bar{x}_j], \) and \( 1 \leq m, n < \infty \). The mathematical operation rules of general grey numbers are

\[
\otimes x_1 + \otimes x_2 \in \bigcup_{i,j}^{m,n} \left[ x_i + x_j, \bar{x}_i + \bar{x}_j \right] 
\]

\[
\otimes x_1 - \otimes x_2 \in \bigcup_{i,j}^{m,n} \left[ x_i - x_j, \bar{x}_i - \bar{x}_j \right] 
\]

A grey number is only one way to model uncertainty, and can be usefully compared to a probability distribution and a fuzzy number. In Fig. 2, the left panel represents a probability distribution on \([0, 1]\), the middle diagram shows the membership of a fuzzy number with lower bound 0 and upper bound 1, and the right panel represents the grey number \([0, 1]\). Note that the probability distribution contains more information than the fuzzy number, but the probability distribution and the fuzzy number both permit relative comparison of \( x \) and \( y \) for \( 0 \leq x < y \leq 1 \), while the grey number contains no information about such a comparison. Since grey numbers do not consider the distribution of possible values, they can handle decision problems with very limited information.

According to the theory of bounded rationality proposed by Simon [39], in decision-making, rationality of individuals is limited by not having enough information, sufficient ability to cognitively process information, and enough time. Sometimes, DMs do not require a perfect or optimal solution because of lack of time to consider what decision to make. The utilization of a grey number within GMCR makes it suitable to conduct a decision analysis and to provide reasonable suggestions to DMs in a problem situation involving high uncertainty.

III. GREY PREFERENCE STRUCTURE IN THE GRAPH MODEL

In the following subsections, the fundamental concepts of grey preference degree (GPD), grey relative certainty of preference (GRCP), anticipated preference (AP), grey satisfying threshold (GST), and grey unilateral improvement (GUI) are defined. These definitions are analogous, but different, to the corresponding definitions for fuzzy preferences in GMCR [17], [18]. Based on these definitions, the concept of grey preference is incorporated into the graph model methodology, and four basic grey stability definitions in a conflict with multiple DMs are described.

A. Grey GPD

When a DM needs to make a choice between two alternatives, sometimes he may easily make the decision, and strictly prefers one over another. For example, a vegetarian definitely prefers vegetables over meat; however, in some situations, it is hard for the DM to choose. For example, a vegetarian may be not sure whether he will cook tomato or potato for lunch. Then, uncertain preferences can capture a DM’s intuition on how much an alternative is preferred to the other. A grey preference expresses uncertain preferences of DMs in a general way using generalized grey numbers, ranging from 0 to 1. Depending on the degree of uncertainty, a grey preference structure allows DMs to represent their preferences in different forms flexibly. For example, a DM can show his preference of one alternative over another as a value, 0.6, an interval, \([0.2, 0.4]\), or a combination of intervals, \([0.3, 0.4, 0.5, 0.6]\). The elements in these grey numbers represent the DM’s possible preference degree for one state over another. Grey preferences constitute an extension of GMCR preference structures.

Definition 14: Let \( D[0, 1]^\otimes \) represent the set of all grey numbers within the interval \([0, 1]\). A grey preference is an \( m \times m \) matrix of grey numbers, \( \otimes P = (\otimes p_{ij})_{m \times m} \), denoted as

\[
\otimes p(s_i, s_j) = \otimes p_{ij} \in D[0, 1]^\otimes 
\]

the GPD of state \( s_i \) over \( s_j \) satisfies \( \otimes p_{ij} = 0.5 \) for \( i = 1, 2, \ldots, m \), and if \( \otimes p_{ij} \in \bigcup_{i=1}^{l} [p_{ij}', \bar{p}_{ij}'] \), then \( \otimes p_{ij} \in \bigcup_{i=1}^{l} [1 - \bar{p}_{ij}', 1 - p_{ij}'] \) for \( i, j = 1, 2, \ldots, m \). The grey-based preference degrees provided by Definition 14 can be interpreted as follows.

1) \( \otimes p(s_i, s_j) = 0 \) indicates state \( s_j \) is strictly preferred to \( s_i \).
2) $\otimes p(s_i, s_j) \in D[0, 0.5]$ and $\otimes p(s_i, s_j) \neq 0, 0.5$ indicate state $s_i$ is less likely to be preferred to $s_j$.

3) $\otimes p(s_i, s_j) = 0.5$ means state $s_i$ is equally likely to be preferred to $s_j$.

4) $\otimes p(s_i, s_j) \in D[0.5, 1]$ and $\otimes p(s_i, s_j) \neq 0, 1$ indicate state $s_j$ is more likely to be preferred to $s_i$.

5) $\otimes p(s_i, s_j) = 1$ indicates state $s_j$ is strictly preferred to $s_i$.

Then, the grey preferences of DM $k$ over all possible pairs of states in $S$ can be represented by a grey preference matrix $(\otimes p^k)_{m \times m}$, generated through pairwise comparison among all possible pairwise combinations of states, and written as follows:

$$\otimes p^k = \begin{bmatrix}
\otimes p_{11}^k & \otimes p_{12}^k & \cdots & \otimes p_{1m}^k \\
\otimes p_{21}^k & \otimes p_{22}^k & \cdots & \otimes p_{2m}^k \\
\vdots & \vdots & \ddots & \vdots \\
\otimes p_{m1}^k & \otimes p_{m2}^k & \cdots & \otimes p_{mm}^k
\end{bmatrix}. \quad (7)$$

An entry in $\otimes p^k$ indicates DM $k$’s preference degree for the row state over the column state.

Three examples are provided so as to further interpret grey preference structure.

**Example 2:** A brownfield property is for sale, and the government (G) has two options: offer financial and policy incentives ($O$) or not ($N$). From its own perspective, the government would prefer the option of $N$. However, successful redevelopment of the brownfield can increase tax revenue and employment opportunities, and remove the potential threat of pollution. The preference for $N$ over $O$ is $\otimes p^{G}_{NO} = 0.8$. This uncertain preference means the government does not definitely prefer $N$ or $O$, but it is more likely to prefer $N$. Then, the preference matrix can be written as

$$\otimes p^G = \frac{O}{N} \begin{bmatrix}
0.5 & 0.2 \\
0.8 & 0.5
\end{bmatrix}. \quad (8)$$

**Example 3:** After a period of time, the property has not been sold, and citizens ask for environmental remediation. Considering the potential income after the property redevelopment and the possible social impacts, the government’s preference over these two states has changed, and its preferences move from $N$ toward $O$. However, it is hard to estimate the magnitude of its preferences change. Therefore, the uncertain preferences are represented using interval values. Compared with the preferences in Example 2, interval preferences mean that $N$ is preferred over $O$ for the government in general, but the degree of preference is not sure. The preference matrix for the government becomes

$$\otimes p^G = \frac{O}{N} \begin{bmatrix}
0.5 & [0.2, 0.4] \\
[0.6, 0.8] & 0.5
\end{bmatrix}. \quad (9)$$

**Example 4:** Later, it is clear that no buyer wants to purchase this property because of the potential risk of pollution, which may lead to endless liability regarding the cleanup of hazardous materials. An overall assessment of onsite contamination is undertaken. The government is eager to facilitate the transaction as soon as possible, and it is reluctant to wait for the assessment results. In this situation, considering the degree of pollution, its preferences of $N$ over $O$ are split into two parts. If the property is highly contaminated the preference of $N$ over $O$ is $[0.3, 0.5]$; but if the property is lightly contaminated the preference of $N$ over $O$ is $[0.6, 0.8]$. Then the preference matrix is

$$\otimes p^G = \frac{O}{N} \begin{bmatrix}
0.5 & [0.2, 0.4], [0.5, 0.7] \\
[0.3, 0.5], [0.6, 0.8] & 0.5
\end{bmatrix}. \quad (10)$$

In modeling a conflict, different forms of grey number can be used to grasp the intuitions of DMs in comparing alternatives according to the degree of uncertainty, especially when information is limited and the options listed in the model do not cover all the concerns of DMs. Accordingly, it is meaningful for DMs to express their uncertain preferences using grey numbers.

Compared with the other methodologies dealing with uncertainty, grey preferences constitute a more suitable way for DMs to express their uncertain preferences, especially at the beginning of a conflict. During that period, DMs in the conflict have limited information about the preferences of the other DMs. Hence, there may be high preference uncertainty.

**B. GRCP**

In a graph model, a grey preference reflects preference uncertainty using general grey numbers, indicating the GPD to which a given state is preferred over another. As the GPD of state $s_i$ over $s_j$ is $\otimes p(s_i, s_j)$, then $\otimes p(s_i, s_j)$ is a measure of the GPD to which state $s_i$ is not preferred to state $s_j$. Then, the GRCP represents the intensity of preference of one state over another.

**Definition 15:** Let $\otimes p^k(s_i, s_j)$ represent the GPD of state $s_i$ over $s_j$ of DM $k \in N$, and $D[-1, 1] \otimes$ represent the set of all grey numbers within the interval $[-1, 1]$. The GRCP for DM $k$ of state $s_i$ relative to $s_j$ is

$$\otimes r^k(s_i, s_j) = \otimes p^k(s_i, s_j) - \otimes p^k(s_j, s_i). \quad (8)$$

In (8), $\otimes r^k(s_i, s_j) \in D[-1, 1] \otimes$. To assist in further interpretation, the following properties are provided. For DM $k$:

1) $\otimes r^k(s_i, s_j) = -1$ indicates state $s_j$ is strictly preferred to $s_i$. A simplified notation for $\otimes r^k(s_i, s_j)$ is $\otimes r^k_{ij}$. Then, a GRCP for DM $k$ over $S$ is represented by a matrix $[\otimes r^k_{ij}]_{m \times m}$.

2) $\otimes r^k(s_i, s_j) \in D[-1, 0] \otimes$ and $\otimes r^k(s_i, s_j) \neq -1, 0$ indicates state $s_j$ is less likely to be preferred to $s_i$.

3) $\otimes r^k(s_i, s_j) = 0$ indicates state $s_j$ is equally likely to be preferred to $s_i$.

4) $\otimes r^k(s_i, s_j) \in D[0, 1] \otimes$ and $\otimes r^k(s_i, s_j) \neq 0, 1$ indicates state $s_j$ is more likely to be preferred to $s_i$.

5) $\otimes r^k(s_i, s_j) = 1$ indicates state $s_j$ is strictly preferred to $s_i$.

Then, the grey relative preferences of DM $k$ over all possible pairs of states in $S$ can be represented by a grey relative preference matrix $\otimes r^k$. Note that because $\otimes r^k(s_i, s_j) = \otimes p^k(s_i, s_j) - \otimes p^k(s_j, s_i) = -\otimes p^k(s_j, s_i) - \otimes p^k(s_i, s_j) = -r^k(s_j, s_i)$), the matrix (9) follows the symmetry property:

$$\otimes r^k = \begin{bmatrix}
\otimes r^k_{11} & \otimes r^k_{12} & \cdots & \otimes r^k_{1m} \\
\otimes r^k_{21} & \otimes r^k_{22} & \cdots & \otimes r^k_{2m} \\
\vdots & \vdots & \ddots & \vdots \\
\otimes r^k_{m1} & \otimes r^k_{m2} & \cdots & \otimes r^k_{mm}
\end{bmatrix}. \quad (9)$$

In modeling a conflict, different forms of grey number can be used to grasp the intuitions of DMs in comparing alternatives according to the degree of uncertainty, especially when information is limited and the options listed in the model do not cover all the concerns of DMs. Accordingly, it is meaningful for DMs to express their uncertain preferences using grey numbers.

Compared with the other methodologies dealing with uncertainty, grey preferences constitute a more suitable way for DMs to express their uncertain preferences, especially at the beginning of a conflict. During that period, DMs in the conflict have limited information about the preferences of the other DMs. Hence, there may be high preference uncertainty.
Example 5: Based on the preferences of Example 3, the GRCP of the government is
\[
\otimes P^G = \frac{O}{N} \begin{bmatrix} 0.0 & 0 \\ [0.2, 0.6] & [-0.6, -0.2] \end{bmatrix} 0.0
\]

A transformation that has been introduced in Definition 15 to convert GPDs to grey relative preference degrees, can assist DMs to identify more easily the relative preferences over the states. After the transformation, for \( s_i, s_j \in S \), a positive value means that \( s_i \) is more likely to be preferred than \( s_j \), and a negative value indicate that \( s_j \) is more likely to be preferred than \( s_i \).

C. GUI

In GMCR, one of the main objectives is to determine whether a DM would prefer to move from one state to another. To identify states that are worthwhile to move to for a DM, two key factors are defined in this section: 1) AP and 2) GST.

In this methodology, APs of DMs on feasible states are determined by characteristics of DMs. The characteristics of human beings, referring in this research to optimism, pessimism, and neutral, have been studied for several decades [40]–[44]. These concepts can be interpreted in a decision process as reflecting how DMs respond differently to the same decision context. Optimistic DMs always hold a positive attitude and anticipate the most desirable outcomes taking place; while pessimistic DMs perceive situations negatively, and thereby think that less preferred outcomes may occur; and neutral DMs believe that an outcome having a middle level of preference will be the result [44].

Note that GRCP, introduced in Definition 15, is represented in the form of a general grey number. Then, AP is employed to estimate the preference of a DM expressed by GRCP, based on three types of characteristics of DMs: 1) optimistic; 2) pessimistic; and 3) neutral.

Definition 16: For \( k \in N \), \( s_i, s_j \in S \), let \( \otimes \rho^k(s_i, s_j) \in \bigcup_{j=1}^n [x_j, \bar{x}_j] \) denote the GRCP of \( s_i \) relative to \( s_j \) for DM \( k \), and let \( r_{ijk}^k \) and \( \bar{r}_{ijk}^k \) represent the lower bound and the upper bound of \( \otimes \rho^k(s_i, s_j) \in \bigcup_{j=1}^n [x_j, \bar{x}_j] \), respectively. Then, the DM \( k \)'s AP for \( s_i \) over \( s_j \), denoted as \( AP^k(s_i, s_j) \), is as follows.

1) If DM \( k \) is pessimistic, then
\[
AP^k(s_i, s_j) = r_{ijk}^k. \tag{10}
\]
2) If DM \( k \) is optimistic, then
\[
AP^k(s_i, s_j) = \bar{r}_{ijk}^k. \tag{11}
\]
3) If DM \( k \) is neutral, then
\[
AP^k(s_i, s_j) = \begin{cases} \frac{1}{n} \sum_{l=1}^n x_l, & \text{if } s_j = \bar{x}_l \text{ for all } l = 1, 2, \ldots, n \\ \frac{\sum_{l=1}^n (x_l - y_l) (y_l + y_l)}{\sum_{l=1}^n (y_l + y_l)} & \text{otherwise.} \end{cases} \tag{12}
\]

To assist in understanding these concepts, note the following.

1) Uncertain preferences over feasible states are represented with general grey numbers, which may consist of a set of intervals. A discrete value is the special case, in which the upper and lower bounds are the same.

2) A pessimistic DM, holding a negative attitude, construes AP as the lower bound of GRCP.

3) An optimistic DM, having a positive attitude, interprets AP as the upper bound of GRCP.

4) The AP of a neutral DM is the center of its GRCP. It can be further interpreted in two forms: 1) the average of the values when they are all discrete and 2) the calculation of AP depends on the midpoint and the length of each interval when intervals are present. Discrete values are not used in this calculation, since they are intervals having no width.

Definition 17: For \( k \in N \) and \( s, s_i \in S \), let \( AP^k(s_i, s) \) denote the AP of DM \( k \) for state \( s_i \) over \( s \). Let \( \gamma_k \) be a real number such that \( AP^k(s_i, s) \geq \gamma_k \) implies DM \( k \) would prefer to move from state \( s \) to \( s_i \). Then \( \gamma_k \) is called the GST of DM \( k \).

Note that \( 0 < \gamma_k \leq 1 \). The GST of a DM is the degree of confidence that characterizes whether the DM finds a move worthwhile. Note that DMs may have different GSTs. In a grey-based preference structure, the decision to move or not is made by analyzing the AP of the target state over the initial state. A DM will move from the initial state only to a state for which the AP is greater or equal to the DM’s GST. For example, an aggressive DM may have a GST of 0.3, but a conservative DM may have a GST of 0.7. The latter DM would move to a state only when its AP over the initial state is less than 0.7.

Since GRCP, AP, and GST have been standardized, a DM’s GUI is formally defined below.

Definition 18: For \( k \in N \) and \( s \in S \), let \( \gamma_k \) be the GST for DM \( k \). Recall that \( R_k(s) \) denotes the set of states reachable from the state \( s \) for DM \( k \). A state \( s_i \in R_k(s) \) is called a GUI from \( s \) for DM \( k \) with respect to \( \gamma_k \), if and only if \( AP^k(s_i, s) \geq \gamma_k \).

A GUI is a preferred state that is reachable by a DM from the initial state. Specifically, a GUI is a reachable state for which AP over the initial state is greater than or equal to the GST of the DM.

Definition 19: For \( s \in S \) and \( k \in N \), let \( R_k(s) \) denote the set of states reachable from the state \( s \) by DM \( k \), and \( \gamma_k \) be the GST for DM \( k \). The GUI list, denoted \( \otimes R^*_k, \gamma_k \), is the collection of all GUIs from \( s \) for DM \( k \) with respect to \( \gamma_k \), represented mathematically as
\[
\otimes R^*_k, \gamma_k(s) = \left\{ s_i \in R_k(s) : AP^k(s_i, s) \geq \gamma_k \right\}. \tag{13}
\]

When more than two DMs are involved in a conflict, joint unilateral improvements for two or more DMs must be taken into account. The unilateral improvement list for \( n \) (\( n > 1 \)) DMs from a given state represents a collection of all possible states to which some or all of the DMs can move via a legal sequence of movements, and each movement is a GUI. A legal sequence of grey moves means that the same DM may move more than once, but not twice consecutively.

Definition 20: For \( s \in S \), \( H \subseteq N \), and \( H \geq 2 \), let \( H = \{1, 2, \ldots, h\} \), and \( \gamma_H = \{\gamma_1, \gamma_2, \ldots, \gamma_n\} \) represent a set of GSTs for corresponding DMs. Let \( \Omega_H^*(s, s_i) \) denote the set of all last DMs in legal sequences allowable for unilateral
improvement from \( s \) to \( s_j \). Then, the GUIs list \( \otimes R^+_{H,\gamma_H}(s) \) from state \( s \) for \( H \) is defined inductively as follows.

1) If \( k \in H \), and \( s_1 \in \otimes R^+_{k,\gamma_k}(s) \), then \( s_1 \in \otimes R^+_{H,\gamma_H}(s) \) and \( k \in \Omega^+_H(s, s_1) \).

2) If \( s_1 \in \otimes R^+_{H,\gamma_H}(s) \), \( k \in H \), \( s_2 \in \otimes R^+_{k,\gamma_k}(s_1) \), and \( \Omega^+_H(s, s_1) \neq \{k\} \), then \( s_2 \in \otimes R^+_{H,\gamma_H}(s) \) and \( k \in \Omega^+_H(s, s_2) \).

Note that the definition stops only when no new state can be added. A joint GUI from a given state by multiple DMs is a state that is in the reachable list for these DMs from the initial state and worthwhile to move to for some or all of the DMs. Specifically, if a group of DMs, \( H \), moves the conflict from state \( s_1 \) to \( s_2 \) via a legal sequence of moves and each movement is a GUI for a corresponding DM according to Definition 18, then \( s_2 \) is a GUI for \( H \), as well as other movements. The unilateral improvement list for multiple DMs is the collection of all GUIs from the given state for any nonempty subset of the DMs.

IV. GREY STABILITIES IN CONFLICT WITH TWO DMs

In GMCR, stability analysis aims to identify stable states for DMs participating in a strategic interaction. The initial state, stable or not, is called the status quo. The DM who has the right to move is called the focal DM. In a graph model with grey preferences, a GUI for a DM is a state to which the DM wishes to move. However, sanctions may be imposed by the other DM, and the focal DM may end up at less preferred states compared with the initial state. In this case, the initial state is stable.

In this section, the four basic grey-based stabilities in a strategic conflict are defined for a graph model with two DMs. Specifically, grey Nash stability (GR), grey general metarationality (GGMR), grey symmetric metarationality (GSMR), and grey SEQ (GSEQ) are introduced. Note that these definitions depend on the grey preference structure that was described in Section III.

Recall that \( S = \{s_1, s_2, \ldots, s_m\} \), \( m > 1 \) denotes the set of feasible states, and \( N = \{p, q\} \) represents the set of two DMs. Then, GSTs for each DM, the APS for \( s_i \) over \( s_j \), and GUIs from \( s \) of each DM are, respectively, denoted as \( \gamma_p \) and \( \gamma_q \), \( AP^p(s_i, s_j) \) and \( AP^q(s_i, s_j) \), and \( \otimes R^+_{p,\gamma_p}(s) \) and \( \otimes R^+_{q,\gamma_q}(s) \). The formal definitions of the four grey stabilities are given below.

Definition 21: A state \( s \in S \) is grey Nash stable or grey rational for DM \( p \), denoted by \( s \in S^p_{GR} \), if and only if \( \otimes R^+_{p,\gamma_p}(s) = \emptyset \).

In GR stability, the focal DM will definitely move to a more preferred state based on his AP and GST without considering possible subsequent countermoves by the other DM. In other words, for the focal DM, no state reachable from the initial state is more preferred, based on his satisfying criterion. Specifically, a state \( s \in S \) is GR stable for DM \( p \) if and only if \( \otimes R^+_{p,\gamma_p}(s) \) has no GUI from \( s \).

Theorem 1: Suppose that for every DM \( k \), \( \otimes R^+_{k,\gamma_k}(s_j, s_j) \), the grey relative preference of DM \( k \) for \( s_j \) over \( s_j \) equals either 1 or \(-1\), for all states \( s_i, s_j \in S \). Then, a state \( s \in S \) is grey Nash stable iff \( s \) is Nash stable in classical GMCR.

Proof: Let \( N = \{p, q\} \) be the set of DMs, and let \( \gamma_p \) denote the GST of \( p \). Let \( \otimes R^+_{p,\gamma_p}(s) = \{s_j \in R_p(s) : AP^p(s_j, s) \geq \gamma_p\} \) represent the GUI list from state \( s \) for DM \( p \). If \( \otimes R^+_{p,\gamma_p}(s) \) is \{1, \(-1\)\}, then \( AP^p(s_j, s) = r^p_j = \bar{r}^p_j = \otimes R^+_{p,\gamma_p}(s) \), \( s \in \{1, \(-1\)\}. Therefore, \( \otimes R^+_{p,\gamma_p}(s) = \emptyset \) if and only if \( \otimes R^+_{p,\gamma_p}(s) = \emptyset \).

Remark 1: Theorem 1 extends to the following three grey stability definitions. Therefore, stability definitions of classical GMCR are special cases of grey-based GMCR.

Definition 22: A state \( s \in S \) is grey GMR for DM \( p \), denoted by \( s \in S^p_{GMR} \), if and only if for every \( s_1 \in \otimes R^+_{p,\gamma_p}(s) \) there exists at least one \( s_2 \in R_p(s_1) \) such that \( AP^p(s_2, s) < \gamma_p \).

In GMGR stability, the focal DM needs to consider only his possible GUIs but also subsequent unilateral movements of the other DM. Specifically, a state \( s \) is GGMR for DM \( p \) if and only if moving to any GUI from \( s \) by DM \( p \) can be sanctioned by a subsequent unilateral movement of DM \( q \). In other words, if \( p \) chooses to move from \( s \) to a GUI, \( s_1, DM q \) has at least one unilateral movement from state \( s_1 \) to a state \( s_2 \), which is less preferred by \( DM p \) compared with \( s \), based on his satisfying criterion.

Definition 23: A state \( s \in S \) is grey SMR stable for DM \( p \), denoted by \( s \in S^p_{SMR} \), if and only if for every \( s_1 \in \otimes R^+_{p,\gamma_p}(s) \) there exists at least one \( s_2 \in R_p(s_1) \) such that \( AP^p(s_2, s) < \gamma_p \), and \( AP^p(s_3, s) < \gamma_p \) for all \( s_3 \in R_p(s_2) \).

In GSMR stability, the focal DM needs to consider not only his possible GUIs but also subsequent unilateral movements of the other DM, as well as the focal DM’s possible counterreactions. Specifically, a state \( s \) is GSMR for DM \( p \) if and only if moving to any GUI from \( s \) by DM \( p \) can be sanctioned by a subsequent unilateral movement of DM \( q \), and DM \( p \) cannot escape this sanction by another unilateral movement. In other words, if DM \( p \) chooses to move to a GUI from \( s \), DM \( q \) has a subsequent unilateral movement from state \( s_1 \) to \( s_2 \), which is not advantageous for DM \( p \) to move to from \( s \), and so is any unilateral movement of DM \( p \) from \( s_2 \), based on his satisfying criterion.

Theorem 2: If state \( s \in S \) is grey SMR stable for DM \( p \), then \( s \) is grey GMR stable for DM \( p \).

Proof: For \( N = \{p, q\} \), let \( R_p(s) \) represent the reachable list from state \( s \) for DM \( p \), and let \( \otimes R^+_{p,\gamma_p}(s) \) be defined as for DM \( p \) if and only if \( s_1 \in \otimes R^+_{p,\gamma_p}(s) \). Then, for any \( s_1 \in \otimes R^+_{p,\gamma_p}(s) \) there exists at least one \( s_2 \in R_p(s_1) \) such that \( AP^p(s_2, s) < \gamma_p \). Hence, \( s \) is grey GMR for DM \( p \).

Remark 2: GSMR adds a restriction to GMGR. Therefore, if \( s \) is GSMR, it must also be GMGR. GMGR is used to define a noncooperative situation, where the opponent may take actions to sanction the focal DM’s improvement without considering his own preferences. GSMR is applicable for DMs having strategic foresight. The focal DM needs to consider not only whether his improvement will be sanctioned by the opponent’s countermoves, but also whether he can escape from this sanction.

Definition 24: A state \( s \in S \) is grey sequentially stable for DM \( p \), denoted by \( s \in S^p_{GSEQ} \), if and only if for every \( s_1 \in \otimes R^+_{p,\gamma_p}(s) \) there exists at least one \( s_2 \in R^+_{q,\gamma_q}(s_1) \) such that \( AP^p(s_2, s) < \gamma_p \).
In GSEQ stability, the focal DM needs to consider not only his possible GUIs but also subsequent GUIs of the other DM. Specifically, a state $s$ is GSEQ stable for DM $p$, if and only if moving to any GUI from $s$ by DM $p$ can be sanctioned by a subsequent GUI of DM $q$. Stated differently, if DM $p$ moves to a GUI $s_1$ from state $s$, DM $q$ has at least one GUI from state $s_1$ to $s_2$, which is less preferred for DM $p$ compared with $s$, based on his satisficing criterion.

**Theorem 3:** If state $s \in S$ is grey sequentially stable for DM $k$, then $s$ is grey GMR stable for DM $k$.

**Proof:** For $N = \{p, q\}$, let $R_p(s)$ represent the reachable list from state $s$ for DM $p$, and let $\otimes R^+_p(s) = \{s_1 \in R_p(s) : AP^p(s_1, s) \geq \gamma_p\}$ stand for the GUI list from state $s$ for DM $p$. If $s$ is grey sequentially stable for DM $p$, then for every $s_1 \in \otimes R^+_p(s)$ there exists at least one $s_2 \in R^+_k(s_1)$ such that $AP^k(s_2, s) < \gamma_p$. Note that $\otimes R^+_q(s_1) \subseteq R_q(s_1)$, that is, for every $s_1 \in \otimes R^+_q(s)$ there exists at least one $s_2 \in R^+_q(s_1)$ such that $AP^q(s_2, s) < \gamma_p$. Hence, $s$ is grey GMR stable for DM $p$.

**Remark 3:** In general, because every unilateral improvement is also an unilateral movement, GSEQ implies GGMR. Hence, if $s$ is GSEQ, it will also be GGMR. Compared with GGMR, GSEQ is suitable when the opponent only considers sanctioning the focal DM’s improvement when he can benefit from the countermove.

**Theorem 4:** If a state $s \in S$ is grey Nash stable for DM $k$, then $s$ is also grey GMR stable, grey SMR stable, and grey sequentially stable for DM $k$.

**Proof:** For $N = \{p, q\}$, let $R_p(s)$ represent the reachable list from state $s$ for DM $p$, and let $\otimes R^+_p(s) = \{s_1 \in R_p(s) : AP^p(s_1, s) \geq \gamma_p\}$ stand for the GUI list from state $s$ for DM $p$. If $s$ is grey Nash stable for DM $p$, then $\otimes R^+_p(s) = \emptyset$. Hence, the Definitions 22–24 are satisfied.

**Remark 4:** If a state $s$ is GR for DM $p$, then the DM does not have any GUI from $s$ to which to move. This also means that no GUI from $s$ by DM $p$ can be sanctioned by the opponent using any unilateral movement or GUI. Finding the relations among the stability definitions is helpful for interpreting a conflict in the process of stability analysis.

**Definition 25:** A state $s \in S$ is called a grey equilibrium under a specific grey stability definition if and only if the state is grey stable for each DM under that grey stability definition.

## V. GREY STABILITIES IN CONFLICT WITH $n$ DMs ($n \geq 2$)

In this section, the four basic grey-based stabilities in a strategic conflict are defined for a graph model having two or more DMs. These definitions depend on unilateral moves controlled by DMs, GUIs, GSTs, characteristics of DMs, and their corresponding APs. Note that $S = \{s_1, s_2, \ldots, s_m\}$, $m > 1$ denotes the set of feasible states and $N$ represents the set of DMs. The formal definitions of the four grey stabilities are given below.

**Definition 26:** A state $s \in S$ is grey Nash stable or grey rational for DM $k$, given by $s \in S^*_{GR_k}$, if and only if $\otimes R^+_{k, GR}(s) = \emptyset$.

If there is no state that is reachable from the initial state or worthwhile for a DM to move to based on its characteristics and satisficing criterion, then the state is GR. In particular, a state $s \in S$ is GR stable for DM $k$ if and only if the DM has no GUI from $s$.

**Definition 27:** A state $s \in S$ is grey GMR for DM $k$, denoted by $s \in S^+_{GR_k}$, if and only if for every $s_1 \in \otimes R^+_{k, GR}(s)$ there exists at least one $s_2 \in R^+_N(s_1)$ such that $AP^k(s_2, s) < \gamma_N$.

If DM $k$ chooses to move from $s$ to a GUI $s_1$, and the other DMs, $N - \{k\}$, have at least one unilateral movement from state $s_1$ to a state $s_2$, which is less preferred for DM $k$ than $s$, then based on his preference, characteristics and satisficing criterion, the GUI from $s$ to $s_1$ for DM $k$ is blocked. If every GUI from $s$ by DM $k$ can be sanctioned by some or all the other DMs’ unilateral movements, then the state is GGMR for DM $k$.

**Definition 28:** A state $s \in S$ is grey sequentially stable for DM $k$, denoted by $s \in S^+_{SEQ_k}$, if and only if for every $s_1 \in \otimes R^+_{k, GR}(s)$ there exists at least one $s_2 \in R^+_N(s_1)$ such that $AP^N(s_2, s) < \gamma_N$. 

If DM $k$ chooses a GUI $s_1$ from state $s$ to which to move, and the other DMs, $N - \{k\}$, have at least one GUI from state $s_1$ to $s_2$, which is not worthwhile for DM $k$ to move from $s$ based on his preference, characteristics, and satisficing criterion, then the GUI from $s$ to $s_1$ for DM $k$ is blocked. If every GUI from $s$ by DM $k$ can be blocked by some or all the other DMs using GUIs given in Definition 20, then the state is GSEQ for DM $k$.

**Definition 29:** A state $s \in S$ is grey SMR for DM $k$, denoted by $s \in S^+_{SMR_k}$, if and only if for every $s_1 \in \otimes R^+_{k, GR}(s)$ there exists at least one $s_2 \in R^+_N(s_1) \cup R^+_N(s)$ such that $AP^k(s_2, s) < \gamma_N$, and $AP^N(s_3, s) < \gamma_N$ for all $s_3 \in R^+_N(s_2)$.

If DM $k$ chooses to move to a GUI $s_1$ from $s$, and the other DMs, $N - \{k\}$, have subsequent unilateral movements from state $s_1$ to $s_2$, which is not worthwhile for DM $k$ from $s$ to move to, and neither is any unilateral movement of DM $k$ from $s_2$, based on his preference, characteristics, and satisficing criterion, then the GUI from $s$ to $s_1$ for DM $k$ is blocked. If every GUI from $s$ by DM $k$ can be sanctioned in the manner described above, then state $s$ is GSMR for DM $k$.

**Definition 30:** A state $s \in S$ is called a grey equilibrium under a specific grey stability definition if and only if the state is grey stable for all DMs under that grey stability definition.

Theorems which are similar to Theorems 1–4 have been developed by the authors for the $n$-DM case. They are not shown here because the results are almost identical to the two-DM case and the proofs are similar.

## VI. EXAMPLE: SUSTAINABLE DEVELOPMENT CONFLICT UNDER UNCERTAINTY

### A. Graph Model

The sustainable development conflict is a hypothetical generic conflict originally proposed by Hipel [45]. In this section, grey-based GMCR, is applied to this conflict to formally handle preference uncertainty experienced by one of the two DMs. This application illustrates how the grey-based solution concepts can be used to analyze a conflict with two DMs having uncertain preferences.
The fundamental components for the conflict are summarized as follows [45].

1) **DMs**: The model consists of two groups of DMs: 1) environmental agencies (ENV) and 2) developers (DEV).

2) **Options**: ENV, aiming to meet human needs while protecting the environment from possible harm, can be proactive or reactive in monitoring the activities of DEV. DEV can choose sustainable development or unsustainable development. In summary, the options for ENV are to be: 1) proactive or 2) reactive, while the options for DEV are to practice: 1) sustainable development or 2) unsustainable development.

3) **Feasible states**: Four feasible states are identified and listed in Table II, in which “Y” means that the option is chosen and “N” means not selected. There are four states in this conflict, for which a state is formed when each DM chooses a strategy. For example, state s1 is created when ENV is proactive in monitoring the activities of DEV, while DEV selects sustainable development.

The directed graph in Fig. 3 shows available transitions by ENV and DEV in a graph model. In this graph, the nodes represent the four feasible states for ENV and DEV, while the directed arcs, labeled by the DMs, represent their respective unilateral movements.

As mentioned above, in the original sustainable development conflict, uncertain preferences of DMs were not taken into account. However, the DMs involved in the conflict may be uncertain about preferences over the feasible states. Li et al. [16] mentioned that preferences of DEV may be uncertain because they may be influenced by enforcement measures of ENV, and some members of DEV may feel more responsible for environmental protection.

### B. Graph Model With Uncertainty

This research uses grey numbers to represent preferences of the DMs. Based on research for the sustainable development conflict conducted by Li et al. [16], Bashar et al. [17], and Hipel [45], it is reasonable to assume that preferences of ENV over feasible states are all certain, and DEV has preference uncertainty between some states. Moreover, it is assumed that ENV is pessimistic and DEV is neutral [16], [45]. Then, through pairwise comparisons over the four feasible states, grey preference matrices are generated for ENV and DEV as listed in Table III.

Note that for $P_{ENV}$, the preference degrees are single discrete values consisting of 1 or 0, representing certain preferences. In $P_{DEV}$, $\otimes p_{23} \in [0.0, 0.3]$ represents DEV’s preference of sustainable development over unsustainable development when ENV is proactive. This uncertainty is because some members of DEV may feel more responsible for environmental protection [46]. The expression $\otimes p_{23} \in [0.25, 0.45, [0.6, 0.7]]$ indicates DEV’s preference of sustainable development when ENV is reactive over unsustainable development when ENV is proactive. For DEV, this uncertainty is influenced by two main factors. One is that the level of enforcement measures is unknown, and hence preferences are split into two parts. In this situation, the preference of DEV does not change consistently. Specifically, if ENV adopts a higher level of enforcement measures, unsustainable development may be punished heavily. Thus, $s_2$ is less preferred than $s_3$ for DEV. On the contrary, if a lower level of enforcement measures is applied by ENV, possible countermeasures may be taken by DEV to achieve more profit through unsustainable development. Then, $s_2$ is more preferred than $s_3$ for DEV. The other factor is that some members of DEV may not agree with others. For example, some members may prefer sustainable development even when ENV employs a lower level of enforcement measures. This is the reason why each split part is expressed with an interval.

### C. Stability Analysis

Based on grey preference matrices of ENV and DEV, GRCPs of the DMs can be calculated according to (8), and the results are shown in Table IV. Then, according to the characteristics of the DMs, AP for ENV and DEV can be calculated, respectively, using (10) and (12). For grey stability analysis, a GST needs to be entertained for each DM having uncertain preference. To analyze the sustainable development conflict under uncertainty, GSTs for DEV are classified into three ranges—[0.0, 0.06], (0.06, 0.6), and [0.6, 1.0]—based on the APs of DEV, which are $A_{p}^{DEV}(s_2, s_1) = 0.6$ and $A_{p}^{DEV}(s_2, s_3) = 0.06$.

The stability results based on the four stability definitions—GR, GGMR, GSMR, GSEQ—are displayed in Table V. In this table, a state that is stable for ENV or DEV is marked with a (√), and a GE indicates that it is stable for both ENV and DEV under corresponding grey stability definitions.

---

**TABLE II**

**Feasible States For The Sustainable Development Conflict**

<table>
<thead>
<tr>
<th>ENV</th>
<th>Sustainable</th>
<th>Proactive</th>
</tr>
</thead>
<tbody>
<tr>
<td>States</td>
<td>Number</td>
<td>s1 s2 s3 s4</td>
</tr>
<tr>
<td>ENV</td>
<td>Y</td>
<td>Y N N N</td>
</tr>
<tr>
<td>DEV</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**TABLE III**

**Grey Preference Matrices of ENV and DEV**

<table>
<thead>
<tr>
<th>DMs</th>
<th>$P_{ENV}$</th>
<th>$P_{DEV}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$s_1$ $s_2$ $s_3$ $s_4$</td>
<td>$s_1$ $s_2$ $s_3$ $s_4$</td>
</tr>
<tr>
<td>$s_1$</td>
<td>0.5</td>
<td>[0.1, 0.3]</td>
</tr>
<tr>
<td>$s_2$</td>
<td>0.0</td>
<td>0.5</td>
</tr>
<tr>
<td>$s_3$</td>
<td>0.0</td>
<td>1.0</td>
</tr>
<tr>
<td>$s_4$</td>
<td>0.0</td>
<td>0.0</td>
</tr>
</tbody>
</table>

Fig. 3. Graph model of movement for the sustainable development conflict.
D. Insights and Sensitivity Analysis

The findings in Table V provide some insights into the sustainable development conflict under uncertainty. When the GST ranges from 0 to 0.06, only one state, \( s_2 \), is stable. In state \( s_2 \), ENV is proactive, and DEV prefers unsustainable development. This result is exactly the same as the stability analysis conducted by [45] when DEV definitely prefers unsustainable states to sustainable development, and uncertainty is not considered. When the GST takes a value from 0.06 to 0.6, a sustainable development state, \( s_3 \), and an unsustainable state, \( s_2 \), become stable. This result may be caused by the enforcement measures chosen by ENV, or by the environmentally friendly members of DEV. In this situation, it may not be worthwhile for DEV to move from a sustainable state to an unsustainable state. When the GST is increased to the range of [0.6,1.0], state \( s_1 \) becomes a new grey equilibrium, while \( s_2 \) and \( s_3 \) remain equilibria. Potentially, both ENV and DEV may search for a proper balance between the needs for social-economic development and environmental protection.

From the above results, a trend can be identified: as the GST increases, the equilibria change from unsustainable states to sustainable ones. To interpret this trend, notice that the uncertain preference of DEV expressed by grey numbers reflects the higher environmental awareness of DEV members and their sensitivity to the potential enforcement measures levied by DEV. Based on the grey preference matrix for DEV, the majority of members of DEV are profit-driven, and put a higher priority on unsustainable development. Recall that for a GUI, APs must exceed GST. In this paper, the more GST increases, the more members of DEV practising sustainability need to be considered. In other words, when the GST is lower, sustainable development is more preferred. On the other hand, with the increase of the GST, sustainable development will gain more priority. Thus, if ENV can make more members realize the importance of environmental protection, or employ a higher level of enforcement measures against unsustainable development, a win–win relationship between ENV and DEV may be achieved.

In this paper, the characteristic of DEV is assumed to be neutral. To have a better understanding of grey-based GMCR, extra stability analyses are conducted when the characteristic of DEV is pessimistic or optimistic, and the results are shown in Tables VI and VII, respectively. Based on the lower bound and the upper bound of each grey number, representing GRCPs of DEV, the GSTs of the DMs are classified into four ranges: 1) \([0.0,0.4]\); 2) \((0.4,0.5]\); 3) \((0.5,0.8]\); and 4) \((0.8,1.0]\). It is easy to conclude from the results the least stable states can be reached when DEV is optimistic, while the most stable states are available when DEV is pessimistic. These findings are caused by different definitions of AP for DEV. It is clear that, if a state is a GUI when DEV is pessimistic, it must also be a GUI when DEV is neutral or optimistic; and if a state is a GUI when DEV is neutral, it must also be a GUI when DEV is optimistic. As a consequence, if a state is stable when DEV is optimistic, it must also be stable when DEV is neutral or pessimistic; and if a state is stable when DEV is neutral, it must also be stable when DEV is pessimistic.

VII. Conclusion

The proposed methodology of grey-based GMCR forms a solid framework for investigating conflict under uncertainty. Based on the general ideas embedded in the original stability concepts in GMCR, the authors introduced the grey-based GMCR, which allows DMs to express their uncertain preferences in a very general way. Furthermore, in order to define grey-based stability definitions (GR, GGMR, GSMR, and GSEQ), certain concepts (GUI, GST, and AP) are introduced to permit grey preferences to be incorporated into these stability definitions and thereby subsequently determine the stable states and equilibria. Finally, through a detailed stability analysis, it can be discovered how the stability results of a conflict
change when the DMs’ GSTs are adjusted. Then, a DM can understand how the uncertainty can influence the results of the conflict under study, thereby avoiding potential risks caused by countermeasures of other DMs or perhaps leading the conflict in a desirable direction.

A case study of sustainable development with two DMs was carried out to demonstrate this new grey-based approach to possibilistic reasoning. Through the comparison with classical GMCR, the grey-based stability analysis results suggested a trend in how the equilibria change. This trend can be helpful for finding the reality hidden behind the uncertainty, and can also assist DMs to estimate the potential evolution of the conflict. In real-world applications, if one can discover where the source of the uncertainty lies, and determine stability analysis results generated by the grey based graph model, the influence of uncertainty on the conflict may be understood. Therefore, DMs can put appropriate countermeasures in place to influence the evolution of the conflict.

A grey preference structure was incorporated into the graph model, thereby allowing DMs to represent their preferences in a flexible way. However, it is more complicated to calculate grey-based stable states than it is when the preferences are crisp. Moreover, when multiple DMs and more states are included in a model, according to Definition 20, joint moves of DMs must be taken into account, and joint GUIs must be identified. Subsequently, a stability analysis can be carried out based on Definitions 26–29. Therefore, a case study on grey-based GMCR with multiple DMs constitutes important future work. In order to complement this research, coalition analysis and status quo analysis of DMs with uncertain preferences in a conflict could also be investigated.

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APPENDIX A

GLOSSARY

<table>
<thead>
<tr>
<th>AP</th>
<th>Anticipated Preference</th>
</tr>
</thead>
<tbody>
<tr>
<td>DM</td>
<td>Decision Maker</td>
</tr>
<tr>
<td>DEV</td>
<td>Developers</td>
</tr>
<tr>
<td>ENV</td>
<td>Environmental Agencies</td>
</tr>
<tr>
<td>GGM</td>
<td>General Metarationality</td>
</tr>
<tr>
<td>GMCR</td>
<td>Graph Model for Conflict Resolution</td>
</tr>
<tr>
<td>GMR</td>
<td>General Metarationality</td>
</tr>
<tr>
<td>GPD</td>
<td>Grey Preference Degree</td>
</tr>
<tr>
<td>GR</td>
<td>Grey Nash Stability</td>
</tr>
<tr>
<td>GRCP</td>
<td>Grey Relative Certainty of Preference</td>
</tr>
<tr>
<td>GSEQ</td>
<td>Grey Sequential Stability</td>
</tr>
<tr>
<td>GSMD</td>
<td>Grey Symmetric Metarationality</td>
</tr>
<tr>
<td>GST</td>
<td>Grey Satisficing Threshold</td>
</tr>
<tr>
<td>GUI</td>
<td>Grey Unilateral Improvement</td>
</tr>
<tr>
<td>R</td>
<td>Nash Stability</td>
</tr>
<tr>
<td>SEQ</td>
<td>Grey Nash Stability</td>
</tr>
<tr>
<td>SMR</td>
<td>Symmetric Metarationality</td>
</tr>
</tbody>
</table>

REFERENCES

Hanhbin Kuang (S’14–M’15) received the bachelor's degree in industrial engineering and the master's degree in management science and engineering (industrial engineering) from the Nanjing University of Aeronautics and Astronautics, Nanjing, Jiangsu, China, in 2007 and 2010, respectively, and the Ph.D. degree in systems design engineering from the University of Waterloo, Waterloo, ON, Canada, in 2015.

He is currently an Assistant Professor of Information Science and Engineering with Northeastern University, Shenyang, China. His current research interests include expanding the graph model for conflict resolution and multiple criteria decision analysis techniques to handle uncertainty using grey systems theory. He applies the foregoing and other systems methodologies to complex decision problems arising in a range of engineering fields.

Dr. Kuang is a member of the IEEE Systems, Man, and Cybernetics Society, the IEEE Young Professionals, and the IEEE Systems Council.
D. Marc Kilgour received the B.A.Sc. degree in engineering physics, the M.Sc. degree in applied mathematics, and the Ph.D. degree in mathematics, all from the University of Toronto, Toronto, ON, Canada.

He is currently a Professor of Mathematics with the Wilfrid Laurier University, Waterloo, ON, Canada, the Research Director of Conflict Analysis with the Laurier Centre for Military Strategic and Disarmament Studies, Waterloo, and an Adjunct Professor of Systems Design Engineering with the University of Waterloo, Waterloo. He has addressed problems in international security, arms control, environmental management, negotiation, arbitration, voting, fair division, electoral systems, social choice, and coalition formation. He pioneered the development of decision support systems for strategic conflicts and their application to negotiation support. Most of his current research interests are contained in the intersection of mathematics, engineering, and social science. His lifetime publications include six books and about 400 articles in refereed journals, conference proceedings, and edited books. He has coedited the book entitled *Handbook of Group Decision and Negotiation* (Springer, 2010).

Dr. Kilgour was the recipient of several international awards and distinctions. He is currently the President of the INFORMS Section on Group Decision and Negotiation. He has supervised the Ph.D. programs of 19 students. He is active in 12 professional societies.